Some effects of the spatial variance on the Cauchy problem for partial differential equations (Harmonic Analysis and Nonlinear Partial Differential Equations)

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By

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Abstract

Several second order semilinear partial differential equations are derived as semilinear field equations in uniform and isotropic spaces. The roles of spatial variance are studied, and several dissipative or antidissipative properties are remarked. This paper is an announcement of the results in a forthcoming paper by the author.

§ 1. The Einstein equation

In this section, we generalize the Einstein equation for the case of three spatial dimensions and real line elements (see e.g., [2] and [4]) into the case of general dimensions and complex line elements. Although we are based on the classical argument for the former case, we show the details for the completeness of the paper. In the following, Greek letters $\alpha, \beta, \gamma, \cdots$ run from 0 to n, Latin letters $j, k, \ell, \cdots$ run from 1 to n. We use the Einstein rule for the sum of indices of tensors, for example, $T^\alpha{}_\alpha := \sum_{\alpha=0}^n T^\alpha{}_\alpha$ and $T^i{}_i := \sum_{i=1}^n T^i{}_i$. For any C^1-curve C in the complex plane $\mathbb{C}$ connecting a point $A \in \mathbb{C}$ to a point $B \in \mathbb{C}$ parametrized by $z = z(x) \in \mathbb{C}$ for $x \in \mathbb{R}$ with $dz/dx \neq 0$, we note that the integration by parts

$$\int_C \frac{df}{dz}(z)g(z)dz = (fg)(B) - (fg)(A) - \int_C f(z)\frac{dg}{dz}(z)dz$$

holds for any C^1-functions f and g. For real variables $x = (x^0, \cdots, x^n) \in \mathbb{R}^{1+n}$ and arbitrarily fixed real numbers $\omega = (\omega^0, \cdots, \omega^n) \in (-\pi/2, \pi/2]^{1+n}$, we consider complex variables $z = (z^0, \cdots, z^n) \in \mathbb{C}^{1+n}$ parametrized by

$$z^\alpha = e^{i\omega^\alpha} x^\alpha. \tag{1.1}$$

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We put $\partial_\alpha := \partial/\partial z^\alpha = e^{-i\omega^\alpha}\partial/\partial x^\alpha$. We define a $(1+n)$-dimensional manifold $\mathcal{M} := \{z \in \mathbb{C}^{1+n} \mid z^\alpha = e^{i\omega^\alpha}x^\alpha,\ x^\alpha \in \mathbb{R},\ 0 \le \alpha \le n\}$. We consider a bilinear symmetric complex-valued functional $\langle\cdot,\cdot\rangle$ on the vector space spanned by the vectors $\{\partial_\alpha\}_{0\le\alpha\le n}$. We put $g_{\alpha\beta}(z) := \langle\partial_\alpha, \partial_\beta\rangle$. We denote by $(g_{\alpha\beta}(z))$ the matrix whose components are given by $\{g_{\alpha\beta}(z)\}_{0\le\alpha,\beta\le n}$. Put $g(z) := \det(g_{\alpha\beta}(z))$. Let $(g^{\alpha\beta}(z))$ be the inverse matrix of $(g_{\alpha\beta}(z))$. We consider a line element

$$-(cd\tau)^2 = (d\ell)^2 := g_{\alpha\beta}(z)dz^\alpha dz^\beta, \tag{1.2}$$

where τ denotes the proper time and we take the square root of $(cd\tau)^2$ as $-\pi < \arg(cd\tau) \le \pi$. We define dz by

$$dz = dz^0 \wedge \cdots \wedge dz^n := \sum_\sigma \operatorname{sgn}(\sigma)\, dz^{\sigma(0)} \cdots dz^{\sigma(n)},$$

where σ denotes the permutation of $\{0,\cdots,n\}$. For the change of variables x to $y = (y^0,\cdots,y^n) \in \mathbb{R}^{1+n}$ by $y = y(x)$, we consider the complex variables $w = (w^0,\cdots,w^n)$ by $w^\alpha = e^{i\omega^\alpha}y^\alpha$. Then we have $\det(\partial z^\alpha/\partial w^\beta) \in \mathbb{R}$, $(-g(z))^{1/2} = |\det(\partial w^\alpha/\partial z^\beta)|(-g(w))^{1/2}$, and $(-g(w))^{1/2}dw = \operatorname{sgn}\det(\partial w^\alpha/\partial z^\beta)\,(-g(z))^{1/2}dz$ by direct calculations, where $g(w)$ denotes the determinant of $(g_{\alpha\beta}(w))$ with $g_{\alpha\beta}(w) := \langle\partial/\partial w^\alpha, \partial/\partial w^\beta\rangle$, and we take the square root of $-g$ as $-\pi < \arg(-g) \le \pi$. We have the fundamental results

$$\begin{aligned} g_{\alpha\beta}\partial_\gamma g^{\alpha\beta} &= -(\partial_\gamma g_{\alpha\beta})g^{\alpha\beta}, \\ \partial_\gamma g^{\alpha\beta} &= -g^{\alpha\mu}(\partial_\gamma g_{\mu\nu})g^{\nu\beta}, \\ \partial_\gamma g &= g g^{\alpha\beta}\partial_\gamma g_{\alpha\beta} \end{aligned}$$

by direct calculations. For any contravariant tensor T^α, we denote its parallel displacement from z to $z+w$ by $\widetilde{T}^\alpha(z+w) := T^\alpha(z) - \Gamma^\alpha{}_{\beta\gamma}(z)T^\beta(z)w^\gamma$, where $\Gamma^\alpha{}_{\beta\gamma}(z)$ denotes the proportional constant at z. We assume the symmetry condition $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$, and

$$\left(g_{\alpha\beta}\widetilde{T}^\alpha\widetilde{T}^\beta\right)(z+w) = \left(g_{\alpha\beta}T^\alpha T^\beta\right)(z) + O\Big(\sum_{0\le\alpha\le n}(w^\alpha)^2\Big)$$

for any T^α and w^α. Then we have the Christoffel symbol

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right). \tag{1.3}$$

We define the covariant derivative ∇_β for T^α by

$$\nabla_\beta T^\alpha(z) := \lim_{w^\beta\to 0}\frac{T^\alpha(z+\widetilde{w^\beta}) - \widetilde{T}^\alpha(z+\widetilde{w^\beta})}{w^\beta} = \partial_\beta T^\alpha(z) + \Gamma^\alpha{}_{\beta\gamma}(z)T^\gamma(z),$$

where $\widetilde{w^\beta} := (0, \cdots, 0, w^\beta, 0, \cdots, 0)$ whose β-component is w^β and the other components are 0. We note that $\nabla_\gamma g_{\alpha\beta} = 0$ and $\nabla_\gamma g^{\alpha\beta} = 0$ follow from (1.3). In general, we define

$$\nabla_\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots} := \partial_\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots} + \Gamma^\alpha{}_{\delta\varepsilon} T^{\varepsilon\beta\cdots}{}_{\mu\nu\cdots} + \Gamma^\beta{}_{\delta\varepsilon} T^{\alpha\varepsilon\cdots}{}_{\mu\nu\cdots} + \cdots \\ - \Gamma^\varepsilon{}_{\delta\mu} T^{\alpha\beta\cdots}{}_{\varepsilon\nu\cdots} - \Gamma^\varepsilon{}_{\delta\nu} T^{\alpha\beta\cdots}{}_{\mu\varepsilon\cdots} - \cdots$$

for any tensor $T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$. By direct calculations, we have

$$\Gamma^\beta{}_{\alpha\beta} = \partial_\alpha \left(\log(-g)^{1/2} \right), \tag{1.4}$$

$$\nabla_\alpha T^\alpha = \frac{1}{(-g)^{1/2}} \partial_\beta \left((-g)^{1/2} T^\beta \right), \tag{1.5}$$

$$\nabla_\alpha \nabla^\alpha \psi = \frac{1}{(-g)^{1/2}} \partial_\beta \left((-g)^{1/2} g^{\beta\gamma} \partial_\gamma \psi \right) \tag{1.6}$$

for any tensor T^α and any scalar ψ.

We define the Riemann curvature tensor

$$R^\delta{}_{\alpha\beta\gamma} := \partial_\beta \Gamma^\delta{}_{\alpha\gamma} - \partial_\gamma \Gamma^\delta{}_{\alpha\beta} + \Gamma^\delta{}_{\varepsilon\beta} \Gamma^\varepsilon{}_{\alpha\gamma} - \Gamma^\delta{}_{\varepsilon\gamma} \Gamma^\varepsilon{}_{\alpha\beta}$$

which is derived from $R^\delta{}_{\alpha\beta\gamma} T^\alpha = (\nabla_\beta \nabla_\gamma - \nabla_\gamma \nabla_\beta) T^\delta$. We define the Ricci tensor $R_{\alpha\beta} := R^\gamma{}_{\alpha\beta\gamma}$, and the scalar curvature $R := g^{\alpha\beta} R_{\alpha\beta}$. We define the Einstein tensor by $G_{\alpha\beta} := R_{\alpha\beta} - g_{\alpha\beta} R/2$. The change of upper and lower indices is done by $g_{\alpha\beta}$ and $g^{\alpha\beta}$, for example, $G^\alpha{}_\beta := g^{\alpha\gamma} G_{\gamma\beta}$.

Let $\Lambda \in \mathbb{C}$ be a constant, which is called the cosmological constant. Let us consider the variation by $g_{\alpha\beta}$ of the Einstein-Hilbert action $\int_{\mathcal{M}} (R + 2\Lambda)\ (-g)^{1/2}\, dz$. By the definitions of the Ricci tensor and the covariant derivative, and by the symmetry condition $\Gamma^\sigma{}_{\nu\mu} = \Gamma^\sigma{}_{\mu\nu}$, we have

$$\delta R_{\rho\mu} = \nabla_\mu \left(\delta \Gamma^\lambda{}_{\rho\lambda} \right) - \nabla_\lambda \left(\delta \Gamma^\lambda{}_{\rho\mu} \right),$$

where $\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$ denotes the variation of $T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$ by $g_{\alpha\beta}$. Since we have $\delta R = (\delta g^{\alpha\beta}) R_{\alpha\beta} + g^{\alpha\beta} \delta R_{\alpha\beta}$ and $g^{\alpha\beta} \delta R_{\alpha\beta} = \nabla_\beta A^\beta$, where we have put $A^\beta := g^{\alpha\beta} \delta \Gamma^\lambda{}_{\alpha\lambda} - g^{\alpha\lambda} \delta \Gamma^\beta{}_{\alpha\lambda}$, we have

$$\delta(R + 2\Lambda) = (\delta g^{\alpha\beta}) R_{\alpha\beta} + \frac{1}{(-g)^{1/2}} \partial_\gamma \left((-g)^{1/2} A^\gamma \right)$$

by (1.5). Since we have $\delta (-g)^{1/2} = -(-g)^{1/2} g_{\alpha\beta} \left(\delta g^{\alpha\beta} \right)/2$, we obtain

$$\delta \int_{\mathcal{M}} (R + 2\Lambda)(-g)^{1/2} dz = \int_{\mathcal{M}} (G_{\alpha\beta} - \Lambda g_{\alpha\beta}) (-g)^{1/2} \delta g^{\alpha\beta} dz \\ + \int_{\mathcal{M}} \partial_\gamma \left((-g)^{1/2} A^\gamma \right) dz.$$

Since the second term in the right hand side vanishes by the divergence theorem, the Euler-Lagrange equation for the Einstein-Hilbert action is given by $G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0$.

For a stress-energy tensor $T^\alpha{}_\beta$, we define the $(1+n)$-dimensional Einstein equation

$$G^\alpha{}_\beta - \Lambda g^\alpha{}_\beta = \kappa\, T^\alpha{}_\beta, \tag{1.7}$$

where κ is a constant and we assume that κc^4 is independent of c. For the case $n = 3$ and real line elements, the constant κ is called the Einstein gravitational constant which is given by $\kappa = 8\pi\mathcal{G}/c^4$, where $\mathcal{G}$ is the Newton gravitational constant. For the case $n \geq 3$ and complex line elements, we are able to generalize the constant κ to

$$\kappa := \frac{2(n-1)\pi^{n/2}\mathcal{G}}{(n-2)\Gamma(n/2)c^4}, \tag{1.8}$$

where Γ denotes the gamma function. Let us show the derivation of (1.8). We denote the volume of the unit ball in $\mathbb{R}^n$ by $\Omega_n := 2\pi^{n/2}/n\Gamma(n/2)$. We put $\hat{z} := (z^1, \cdots, z^n)$, $r(\hat{z}) := \left\{\sum_{j=1}^n (z^j)^2\right\}^{1/2}$, and $\omega^1 = \cdots = \omega^n$ in (1.1). We define a function $E(\hat{z})$ by

$$E(\hat{z}) := \begin{cases} \dfrac{1}{(2-n)n\Omega_n} r(\hat{z})^{2-n} & \text{if } n \geq 3, \\ \dfrac{1}{n\Omega_n} \log r(\hat{z}) & \text{if } n = 2, \\ \dfrac{1}{n\Omega_n} r(\hat{z}) & \text{if } n = 1. \end{cases}$$

Since $E(\hat{x})$ for $\hat{x} = (x^1, \cdots, x^n) \in \mathbb{R}^n$ is the fundamental solution of the Laplacian, the function $E(\hat{z})$ satisfies

$$\Delta_{\hat{z}} E(\hat{z}) = \delta(\hat{z}), \tag{1.9}$$

where $\Delta_{\hat{z}} := \sum_{j=1}^n \partial^2/(\partial z^j)^2$ and δ denotes the Dirac δ-function. We assume that $(g_{\alpha\beta})$ is sufficiently close to the Minkowski matrix $(\eta_{\alpha\beta}) := \operatorname{diag}(-c^2, 1, \cdots, 1)$. Namely, we put $h_{\alpha\beta} := g_{\alpha\beta} - \eta_{\alpha\beta}$, and we assume that $|h_{\alpha\beta}|$ is sufficiently small. For a potential $\phi = \phi(\hat{z})$ and the Lagrangian $L(\hat{z}, d\hat{z}/d\tau) := \sum_{j=1}^n (dz^j/d\tau)^2/2 - \phi(\hat{z})$, the Euler-Lagrange equation for the action $\int L(\hat{z}, d\hat{z}/d\tau)d\tau$ is given by

$$\frac{d^2\hat{z}}{d\tau^2} + \nabla_{\hat{z}}\phi = 0, \tag{1.10}$$

where we have put $\nabla_{\hat{z}} := (\partial_1, \cdots, \partial_n)$. We regard this equation as the equation of motion in our setting. Since a natural extension of the Newton equation for a particle at $\hat{z}$ in the gravitational field by K-particles with mass $m(k)$ at $\hat{z}(k)$ for $1 \leq k \leq K$ has the form

$$\frac{d^2\hat{z}}{d\tau^2} = -\sum_{k=1}^K \mathcal{G} \cdot \frac{m(k)}{r(\hat{z} - \hat{z}(k))^{n-1}} \cdot \frac{\hat{z} - \hat{z}(k)}{r(\hat{z} - \hat{z}(k))},$$

we formulate the Newton equation (1.10) by $\phi := n\Omega_n \mathcal{G}\rho *_{\hat{z}} E$, where $\rho = \rho(\hat{z})$ denotes the density of mass. We note that

$$\Delta_{\hat{z}}\phi = n\Omega_n \mathcal{G}\rho \tag{1.11}$$

holds by (1.9). The Euler-Lagrange equation for the action

$$\int \left(-g_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\right)^{1/2} d\tau$$

yields the equation of the geodesic curve as

$$\frac{d^2z^\delta}{d\tau^2} + \Gamma^\delta{}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau} = 0. \tag{1.12}$$

Let us assume $\partial_0 g_{\alpha\beta} \doteqdot 0$, $g^{0k}\partial_k g_{00} \doteqdot 0$, $h^{\alpha\beta}\partial_\gamma h_{\delta\varepsilon} \doteqdot 0$ and $dz^j/dz^0 \doteqdot 0$. Then we have $\Gamma^\lambda{}_{00} \doteqdot -g^{\lambda k}\partial_k g_{00}/2$, $\Gamma^0{}_{00} \doteqdot 0$, $\Gamma^j{}_{00} \doteqdot -\partial_j g_{00}/2$ and $d\tau \doteqdot dz^0$ by the definitions of $\Gamma^\alpha{}_{\beta\gamma}$ and $d\tau$. So that, we have $d^2z^j/(dz^0)^2 + \Gamma^j{}_{00} \doteqdot 0$ which yields

$$\partial_j\left(\phi + \frac{g_{00}}{2}\right) \doteqdot 0 \tag{1.13}$$

for $1 \le j \le n$ by (1.10) and (1.12). Let us consider the case $\Lambda = 0$ in (1.7). We have

$$(n-1)R = -2\kappa T \tag{1.14}$$

and

$$(n-1)R_{\alpha\beta} = \kappa\left((n-1)T_{\alpha\beta} - Tg_{\alpha\beta}\right) \tag{1.15}$$

when $n \ge 1$ by (1.7). Under the assumption $\partial_\alpha h_{\beta\gamma}\partial_\delta h_{\varepsilon\zeta} \doteqdot 0$, we have

$$R_{00} \doteqdot -\partial_j\Gamma^j{}_{00} \doteqdot \frac{1}{2}\Delta_{\hat{z}}g_{00} \doteqdot -\Delta_{\hat{z}}\phi = -n\Omega_n\mathcal{G}\rho, \tag{1.16}$$

where we have used the definition of Ricci tensor, the above fact $\Gamma^j{}_{00} \doteqdot -\partial_j g_{00}/2$, (1.13) and (1.11). We now consider the stress-energy tensor $T^{\alpha\beta}$ given by

$$T^{\alpha\beta} := -\rho\,\frac{\partial z^\alpha}{\partial\tau}\,\frac{\partial z^\beta}{\partial\tau}$$

based on the analogy to the stress tensor of the perfect gas. We have

$$T^{\alpha\beta} \doteqdot \begin{cases} -\rho \text{ if } (\alpha,\beta) = (0,0), \\ 0 \quad \text{if } (\alpha,\beta) \ne (0,0) \end{cases} \tag{1.17}$$

by $\partial z^j/\partial z^0 \doteqdot 0$. So that, we have

$$T \doteqdot -\rho g_{00} \tag{1.18}$$

and

$$T_{\alpha\beta} \doteqdot \begin{cases} -\rho(g_{00})^2 & \text{if } (\alpha,\beta)=(0,0), \\ 0 & \text{if } (\alpha,\beta)\neq(0,0). \end{cases} \tag{1.19}$$

Therefore, we obtain

$$n(n-1)\Omega_n \mathcal{G}\rho \doteqdot (n-2)\kappa\rho(g_{00})^2 \tag{1.20}$$

by (1.15) and (1.16). The required result (1.8) holds when $n \geq 3$ by $g_{00} \doteqdot -c^2$ and (1.20). When $n=2$, we have $T^{\alpha\beta} \doteqdot 0$ since $\rho = 0$ by (1.20). When $n=1$, we have $\kappa T^{\alpha\beta} \doteqdot 0$ since we have $\kappa = 0$ or $\rho = 0$ by (1.20).

§ 2. Robertson-Walker metrics

We put $r := \left(\sum_{j=1}^{n}(z^j)^2\right)^{1/2}$. We assume that the space is uniform and isotropic, and we consider the line element

$$g_{\alpha\beta}dz^\alpha dz^\beta := -c^2(dz^0)^2 + e^{h(z^0)}e^{f(r)}\sum_{j=1}^{n}(dz^j)^2, \tag{2.1}$$

where h and f are complex-valued functions. This line element is uniform in the sense that for any two points P and Q in $\mathbb{C}^n$, the ratio of the coefficients $e^{h(z^0)}e^{f(r(P))}/e^{h(z^0)}e^{f(r(Q))}$ is independent of z^0.

By direct calculations, we have $G^0{}_j = G^j{}_0 = 0$,

$$G^0{}_0 := \frac{n-1}{2c^2}\left\{\frac{n}{4}(\partial_0 h)^2 - c^2 e^{-h-f}\left(f'' + (n-1)\frac{f'}{r} + \frac{n-2}{4}(f')^2\right)\right\},$$

and

$$\begin{aligned} G^j{}_k := g^j{}_k \Bigg\{ & \frac{n-1}{2c^2}\left(\partial_0^2 h + \frac{n}{4}(\partial_0 h)^2\right) \\ & - \frac{n-2}{2}e^{-h-f}\left(f'' + (n-2)\frac{f'}{r} + \frac{n-3}{4}(f')^2\right)\Bigg\} \\ & + \frac{n-2}{2}e^{-h-f}\left(f'' - \frac{f'}{r} - \frac{(f')^2}{2}\right)\frac{z^j z^k}{r^2}, \end{aligned}$$

where $f' := df/dr$. Since the space is isotropic, the coefficient of $z^j z^k$ must vanish. So that, we assume that f satisfies $f'' - f'/r - (f')^2/2 = 0$, by which we obtain

$$e^f = q^2\left(1 + \frac{k^2 r^2}{4}\right)^{-2} \tag{2.2}$$

for constants $q(\neq 0), k \in \mathbb{C}$. We define a function

$$a(z^0) := e^{h(z^0)/2}. \tag{2.3}$$

Let us consider the stress-energy tensor $T^\alpha{}_\beta$ of the perfect fluid

$$T^\alpha{}_\beta := \mathrm{diag}(\rho c^2, -p, \cdots, -p)$$

for constant density ρ and pressure p. We put $\widetilde{\rho} := \rho + \Lambda/\kappa c^2$ and $\widetilde{p} := p - \Lambda/\kappa$. Then (1.7) is rewritten as $G^\alpha{}_\beta = \kappa \cdot \mathrm{diag}(\widetilde{\rho} c^2, -\widetilde{p}, \cdots, -\widetilde{p})$, which shows that the cosmological constant $\Lambda > 0$ is regarded as the energy which has positive density and negative pressure in the vacuum $\rho = p = 0$ for $\kappa > 0$ ("the dark energy" for $n = 3$). The equation $G^0{}_0 = \kappa \widetilde{\rho} c^2 g^0{}_0$ is rewritten as

$$\frac{n-1}{2}\left\{\left(\frac{\partial_0 a}{ca}\right)^2 + \frac{k^2}{q^2 a^2}\right\} = \frac{\kappa c^2}{n} \cdot \widetilde{\rho}. \tag{2.4}$$

The equation $G^j{}_k = -\kappa \widetilde{p} g^j{}_k$ is rewritten as

$$\frac{n-1}{2}\left\{\frac{2}{n-2} \cdot \frac{\partial_0^2 a}{c^2 a} + \left(\frac{\partial_0 a}{ca}\right)^2 + \frac{k^2}{q^2 a^2}\right\} = -\frac{\kappa}{n-2} \cdot \widetilde{p}, \tag{2.5}$$

which is rewritten as the Raychaudhuri equation

$$\frac{\partial_0^2 a}{c^2 a} = -\frac{n-2}{n-1} \cdot \kappa \left(\frac{\widetilde{\rho} c^2}{n} + \frac{\widetilde{p}}{n-2}\right) \tag{2.6}$$

by (2.4). Multiplying a^n to the both sides in (2.4), taking the derivative by z^0 variable, and using (2.5), we have the conservation of mass

$$\partial_0(\widetilde{\rho} c^2 a^n) + \widetilde{p} \partial_0 a^n = 0. \tag{2.7}$$

For any number σ, we assume the equation of state

$$\widetilde{p} = \sigma \widetilde{\rho} c^2. \tag{2.8}$$

Then $a(z^0)$ must satisfy

$$\frac{\partial_0^2 a(z^0)}{c^2 a(z^0)} = -\frac{n-2+n\sigma}{n(n-1)} \cdot \kappa \widetilde{\rho} c^2$$

with

$$\widetilde{\rho} = \frac{n-1}{2} \cdot \frac{n}{\kappa c^4} \cdot \frac{\partial_0 a(0)^2}{a(0)^{2-n(1+\sigma)}} \cdot a(z^0)^{-n(1+\sigma)} \tag{2.9}$$

by (2.6) and (2.7), which has the solution

$$a(z^0) := \begin{cases} a(0)\left(1+\frac{n(1+\sigma)\partial_0 a(0) z^0}{2a(0)}\right)^{2/n(1+\sigma)} & \text{if } \ \sigma \neq -1, \\ a(0)\exp\left(\frac{\partial_0 a(0) z^0}{a(0)}\right) & \text{if } \ \sigma = -1. \end{cases} \tag{2.10}$$

By the above argument, we have derived the line element

$$g_{\alpha\beta} dz^\alpha dz^\beta = -c^2 (dz^0)^2 + a(z^0)^2 q^2 \left(1+\frac{k^2 r^2}{4}\right)^{-2} \sum_{j=1}^{n} (dz^j)^2 \tag{2.11}$$

for constants $q(\neq 0), k \in \mathbb{C}$ as the solution of (1.7). By (2.4), (2.9) and (4.1), we have $k=0$.

§ 3. Field equations

For any $\lambda \in \mathbb{C}$ and any complex-valued C^2 function ϕ on $\mathcal{M}$, we define the Lagrangian

$$L(\phi) := -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2 + \frac{\lambda}{p+1} |\phi|^{p-1} \phi^2.$$

We apply the variational method to the action $\int_{\mathcal{M}} L(\phi)(-g)^{1/2} dz$ for ϕ. Then the Euler-Lagrange equation is given by

$$\frac{1}{(-g)^{1/2}} \partial_\alpha ((-g)^{1/2} g^{\alpha\beta} \partial_\beta \phi) - \left(\frac{mc}{\hbar}\right)^2 \phi + \lambda |\phi|^{p-1} \phi = 0 \tag{3.1}$$

under the constraint condition $\arg \delta\phi = \arg \phi$. This is the equation of motion of massive scalar field described by a function ϕ with the mass m and the potential $\lambda|\phi|^{p-1}\phi^2/(p+1)$.

For example, let us consider the simplest case that the scale-function is a constant $a=1$. From (3.1), we are able to obtain the Klein-Gordon equation

$$\partial_t^2 \phi - c^2 \Delta_x \phi + \frac{m^2 c^4}{\hbar^2} \phi - c^2 \lambda |\phi|^{p-1} \phi = 0, \tag{3.2}$$

the Schrödinger equation

$$\pm i \frac{2m}{\hbar} \partial_t u + \Delta_x u + \lambda |u|^{p-1} u = 0, \tag{3.3}$$

the elliptic equation

$$\partial_t^2 \phi + c^2 \Delta_x \phi + \frac{m^2 c^4}{\hbar^2} \phi - c^2 \lambda |\phi|^{p-1} \phi = 0, \tag{3.4}$$

and the parabolic equation

$$\frac{2m}{\hbar} \partial_t u - \Delta_x u - i\lambda |u|^{p-1} u = 0, \tag{3.5}$$

where we have put $\Delta_x := \sum_{j=1}^n \partial^2/(\partial x^j)^2$, and we use the nonrelativistic limit to derive the Schrödinger equation and the parabolic equation.

§ 4. Cauchy problem

In this section, we research on the effects of the spatial variance $a(\cdot)$. Let $\sigma \in \mathbb{R}$, $a_0 > 0$, $a_1 \in \mathbb{R}$. We put $T_0 := \infty$ when $(1+\sigma)a_1 \geq 0$, $T_0 := -2a_0/n(1+\sigma)a_1 (> 0)$ when $(1+\sigma)a_1 < 0$. We define a scale-function $a(t)$ for $t \in [0, T_0)$ by

$$a(t) := \begin{cases} a_0 \left(1 + \frac{n(1+\sigma)a_1 t}{2a_0}\right)^{2/n(1+\sigma)} & \text{if } \sigma \neq -1, \\ a_0 \exp\left(\frac{a_1 t}{a_0}\right) & \text{if } \sigma = -1, \end{cases} \tag{4.1}$$

where we note that $a_0 = a(0)$ and $a_1 = \partial_t a(0)$. The field equation (3.1) is rewritten as

$$-\frac{1}{c^2}\left(\partial_t^2 + \frac{n\partial_t a}{a}\partial_t + \frac{m^2c^4}{\hbar^2}\right)\phi + \frac{1}{a^2 e^{i2\omega}}\Delta\phi - \frac{\lambda}{e^{2i\omega}}|\phi|^{p-1}\phi = 0 \tag{4.2}$$

for $(t,x) \in [0, T_0) \times \mathbb{R}^n$ with $z^\alpha = e^{i\omega}x^\alpha$, where $\lambda \in \mathbb{C}$, $1 \leq p < \infty$, $-\pi/2 < \omega \leq \pi/2$, and $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$. We define a weight function $w(t) := (a_0/a(t))^{n/2}$. The equation (4.2) yields

$$\pm i\frac{2m}{\hbar}\partial_t u + \frac{1}{a^2 e^{2i\omega}}\Delta u - \frac{\lambda}{e^{2i\omega}}|uw|^{p-1}u = 0 \tag{4.3}$$

by a transformation from ϕ to u, and the nonrelativistic limit ($c \to \infty$), where $i := \sqrt{-1}$ and the double sign $\pm$ is in same order throughout the paper.

Let us consider how the variance affects the existence of the solutions. Since the equation (4.3) has a variable coefficient, we use a change of variable $s = s(t) := \int_0^t a(\tau)^{-2}d\tau$. We put $S_0 := s(T_0)$. We use conventions $a(s) := a(t(s))$ and $w(s) := w(t(s))$ for $s \in [0, S_0)$ as far as there is no fear of confusion. A direct computation shows

$$S_0 = \begin{cases} \frac{2}{a_0 a_1 (4 - n(1+\sigma))} & \text{if } a_1(4 - n(1+\sigma)) > 0, \\ \infty & \text{if } a_1(4 - n(1+\sigma)) \leq 0. \end{cases}$$

For $0 \leq \mu_0 < n/2$ and $0 < S \leq S_0$, we consider the Cauchy problem given by

$$\begin{cases} \pm i\frac{2m}{\hbar}\partial_s u(s,x) + \frac{1}{e^{2i\omega}}\Delta u(s,x) - \frac{\lambda a(s)^2}{e^{2i\omega}}\left(|uw|^{p-1}u\right)(s,x) = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{4.4}$$

for $(s,x) \in [0,S) \times \mathbb{R}^n$, where $H^{\mu_0}(\mathbb{R}^n)$ denotes the Sobolev space of order $\mu_0 \geq 0$. Since $u = u(t,\cdot)$ is a global solution of (4.3) if it exists on $[0, T_0)$, we say $u = u(s,\cdot) = u(t(s),\cdot)$ is a global solution of (4.4) if it exists on $[0, S_0)$. Here, (4.3) is rewritten as the first equation in (4.4) by the change of variable $s = s(t)$.

Let us consider the well-posedness of (4.4). For any real numbers $2 \le q \le \infty$ and $2 \le r < \infty$, we say that the pair (q, r) is admissible if it satisfies $1/r + 2/nq = 1/2$. For $\mu_0 \ge 0$ and two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$, we define a function space

$$X^{\mu_0}([0,S)) := \{u \in C([0,S), H^{\mu_0}(\mathbb{R}^n));\ \max_{\mu=0,\mu_0} \|u\|_{X^{\mu}([0,S))} < \infty\}$$

with a metric $d(u,v) := \|u-v\|_{X^0([0,S))}$ for u, $v \in X^{\mu_0}([0,S))$, where

$$\|u\|_{X^{\mu}([0,S))} := \begin{cases} \|u\|_{L^\infty((0,S),L^2(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),L^{r_j}(\mathbb{R}^n))} & \text{if } \mu = 0, \\ \|u\|_{L^\infty((0,S),\dot{H}^{\mu}(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),\dot{B}^{\mu}_{r_j 2}(\mathbb{R}^n))} & \text{if } \mu > 0. \end{cases}$$

Here, $\dot{H}^{\mu}(\mathbb{R}^n)$ and $\dot{B}^{\mu}_{r_j 2}(\mathbb{R}^n)$ denote the homogeneous Sobolev and Besov spaces, respectively. Since the propagator of the linear part of the first equation in (4.4) is written as $\exp(\pm i\hbar \exp(-2i\omega)s\Delta/2m)$, we assume $0 \le \pm\omega \le \pi/2$ to define it as a pseudo-differential operator. We note that the scaling critical number of p for (4.4) is $p(\mu_0) := 1 + 4/(n-2\mu_0)$ when $a(\cdot) = 1$. We put

$$p_1(\mu_0) := 1 + \frac{4}{n-2\mu_0} \cdot \left(1 + \frac{4}{n-2\mu_0} \cdot \frac{2\mu_0}{n(1+\sigma)}\right)^{-1}$$

for $\sigma \ne -1$.

Theorem 4.1. *Let $n \ge 1$, $\lambda \in \mathbb{C}$, $0 \le \mu_0 < n/2$, and $1 \le p \le p(\mu_0)$. Let ω satisfies $0 \le \pm\omega \le \pi/2$ and $\omega \ne -\pi/2$. Assume $\mu_0 < p$ if p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, there exist $S > 0$ with $S \le S_0$ and a unique local solution u of (4.4) in $X^{\mu_0}([0,S))$. Here, S depends only on the norm $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ when $p < p(\mu_0)$, while S depends on the profile of u_0 when $p = p(\mu_0)$. The solutions depend on the initial data continuously.

(2) (Small global solutions.) Assume that one of the following conditions from (i) to (vi) holds: (i) $\mu_0 = 0$, $p = p(0)$, (ii) $\mu_0 > 0$, $p = p(\mu_0)$, $a_1 \ge 0$, (iii) $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma < -1$, (iv) $1 < p < p_1(\mu_0)$, $a_1 < 0$, $\sigma > -1$, (v) $p_1(\mu_0) < p < p(\mu_0)$, $a_1 > 0$, $\sigma > -1$, (vi) $\mu_0 > 0$, $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma = -1$. If $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution, namely, $S = S_0$.

When $\lambda \in \mathbb{R}$, we are able to use the conservation law to show global solutions for large data in $H^1(\mathbb{R}^n)$. When $a(\cdot) = 1$, the following results are known. The global solutions were shown for the Schrödinger equation in [5, Theorem 3.1], for the complex Ginzburg-Landau equation in [6, Proposition 4.2]. Blow-up solutions for initial data with negative energy are obtained by the concavity of an auxiliary function, the virial identity, and the Heisenberg uncertainty principle. We refer to [3, Section 6.5] and [1, Theorem 1.8] for Schrödinger equations, and [7, Theorem 5.3] for parabolic equations.

We are able to obtain the following results for global and blow-up solutions for (4.4).

Corollary 4.2. *Let $\mu_0 = 0$ or $\mu_0 = 1$. Let $\lambda > 0$. Let $1 \leq p < 1 + 4/n$ when $\mu_0 = 0$. Let $1 \leq p < 1 + 4/(n-2)$ and $a_1(p - 1 - 4/n) \geq 0$ when $\mu_0 = 1$. For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 4.1 is a global solution.*

Corollary 4.3. *Let $\mu_0 = 1$, $\lambda < 0$, $a_1 \geq 0$ and $1 \leq p < 1 + 4/n$. Let $\omega = 0$ or $\omega = \pi/2$. For any $u_0 \in H^1(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 4.1 is a global solution.*

Corollary 4.4. *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega \neq 0, \pi/2$. Put $p_0 := 2/(\sin 2\omega)^2 - 1$. Let $p_0 < p \leq 1 + 4/(n-2)$. Let $a_1(p - 1 - 4/n) \leq 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ with negative energy*

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_0(x)|^2 + \frac{\lambda a_0^2 |u_0(x)|^{p+1}}{p+1} dx < 0, \tag{4.5}$$

the solution u given by (1) in Theorem 4.1 blows up in finite time.

Corollary 4.5. *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega = 0$ or $\omega = \pi/2$. Let $1 + 4/n \leq p \leq 1 + 4/(n-2)$. Let $a_1 \leq 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ which satisfies $\||x|u_0(x)\|_{L^2_x(\mathbb{R}^n)} < \infty$ and (4.5), the solution u given by (1) in Theorem 4.1 blows up in finite time.*

To prove the above corollaries, we use two dissipative properties. One is from the parabolic structure of the first equation in (4.4) when $0 < \pm\omega < \pi/2$. The other is from the scale function $a(\cdot)$ when $\partial_t a(0) = a_1 \neq 0$. Even if the equation does not have the parabolic structure when $\omega = 0, \pi/2$, the latter is very effective to obtain the global solutions.

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